

Computing the Least-core and Nucleolus for Threshold Cardinality Matching Games

Qizhi Fang¹, Bo Li¹, Xiaoming Sun² *, Jia Zhang², and Jialin Zhang²

¹ School of Mathematical Sciences, Ocean University of China, Qingdao, China

² Institute of Computing Technology, Chinese Academy of Sciences, Beijing, China

Abstract. Cooperative games provide a framework for fair and stable profit allocation in multi-agent systems. *Core*, *least-core* and *nucleolus* are such solution concepts that characterize stability of cooperation. In this paper, we study the algorithmic issues on the least-core and nucleolus of threshold cardinality matching games (TCMG). A TCMG is defined on a graph $G = (V, E)$ and a threshold T , in which the player set is V and the profit of a coalition $S \subseteq V$ is 1 if the size of a maximum matching in $G[S]$ meets or exceeds T , and 0 otherwise. We first show that for a TCMG, the problems of computing least-core value, finding and verifying least-core payoff are all polynomial time solvable. We also provide a general characterization of the least core for a large class of TCMG. Next, based on Gallai-Edmonds Decomposition in matching theory, we give a concise formulation of the nucleolus for a typical case of TCMG which the threshold T equals 1. When the threshold T is relevant to the input size, we prove that the nucleolus can be obtained in polynomial time in bipartite graphs and graphs with a perfect matching.

1 Introduction

One of the important problems in cooperative game is how to distribute the total profit generated by a group of agents to individual participants. The prerequisite here is to make all the agents work together, *i.e.*, form a grand coalition. To achieve this goal, the collective profit should be distributed properly so as to minimize the incentive of subgroups of agents to deviate and form coalitions of their own. This intuition is formally captured by several solution concepts, such as the *core*, the *least-core*, and the *nucleolus*, which will be the focus of this paper.

The algorithmic issues in cooperative games are especially interesting since the definitions of many solution concepts would involve in an exponential number of constraints [20]. Megiddo [17] suggested that finding a solution should be done by an efficient algorithm (following Edmonds [9]), *i.e.*, within time polynomial in the number of agents. Deng and Papadimitriou [8] suggested the computational complexity be taken into consideration as another measure of fairness for evaluating and comparing different solution concepts. Subsequently,

* Email: sunxiaoming@ict.ac.cn

various interesting complexity and algorithmic results have been investigated. On one hand, efficient algorithms have been proposed for computing the core, the least-core and the nucleolus, such as, for assignment games [22], standard tree games [13], matching games [14], airport profit games [4], spanning connectivity games [2], flow games [6] and weighted voting games [11]. On the other hand, some negative results are given. For example, the problems of computing the nucleolus and testing whether a given distribution belongs to the core or the nucleolus are proved to be NP-hard for minimum spanning tree games [12], flow games and linear production games [6].

Matching game is one of the most important combinatorial cooperative games which has attracted much attention [3, 5, 7, 14, 21, 22]. Assignment game is a special case of matching game defined on a bipartite graph, which is introduced by Shapley and Shubik [21] to formulate two-sided markets. In this case, the core is always non-empty, and the nucleolus can be found in polynomial time [22]. For matching games defined on general graphs, Deng, et al., [7] gave a sufficient and necessary condition on the non-emptiness of the core. Kern and Paulusma [14] presented an efficient algorithm for computing the nucleolus for cardinality matching games based on a polynomial description of the least-core of such games. Recently, Biró, Kern, and Paulusma [3] generalized the work of [22] to develop an efficient algorithm for computing the nucleolus for matching games on weighted graphs when the core is nonempty. Chen, Lu and Zhang [5] further discussed the fractional matching games. However, for the matching games defined on general weighted graphs, the computational complexity on the least-core and the nucleolus is still open.

We follow the stream and study the least-core and nucleolus of a natural variation of matching games, called threshold matching games [1]. In this game model, each agent controls a vertex in an edge-weighted graph, and a coalition wins only if the maximum weighted matching in the induced subgraph meets or exceeds a given threshold value. Although Haris et al [1] proved that computing the least-core and the nucleolus for threshold matching games defined on general weighted graphs is NP-hard, the related algorithmic problems have not been discussed when restricted to unweighted graphs.

In this paper, we aim to compute the least-core and the nucleolus for the threshold matching games on unweighted graph, especially when the core is empty. Firstly, we show that for an arbitrary threshold value, the least-core can be obtained in polynomial time through separation oracle technique. By linear program duality, we further provide a general characterization of the least core for a large class of threshold cardinality matching games, which can be used to simplify the sequence of linear programs of the nucleolus. Secondly, we discuss the algorithms for the nucleolus. In the case that the threshold is independent of input size, the nucleolus can be found in polynomial time. Especially, when the threshold being one (which is called *edge coalitional games*), we know that finding the least-core and the nucleolus can be done efficiently based on a clear description of the least-core. When the threshold value is relevant to the input size, we prove that the least-core and the nucleolus can also be computed in polynomial

time for the games on two typical graphs, the graphs with a perfect matching or bipartite graphs. To our surprise, in all the cases considered, the least-core and the nucleolus do not depend on the value of the threshold. We conjecture our method can be generalized into dealing with general graphs. Besides, we establish the relationship between the least-core of a threshold matching game and the mixed Nash Equilibrium of a non-cooperative two-person zero-sum game, called *matching intercept game*.

The organization of the paper is as follows. In section 2, we introduce some concepts in cooperative game theory, and the definitions of *threshold matching game* (TMG) and *threshold cardinality matching game* (TCMG). In section 3, we discuss the least-core for TCMGs. Section 4, section 5 and section 6 are dedicated to the efficient algorithms for computing the nucleolus of *edge coalitional games* (ECG), *TCMG on graphs with a perfect matching*, and *TCMG on bipartite graphs*.

2 Preliminary and Definition

2.1 Cooperative Game Theory

A cooperative game $\Gamma = (N, v)$ consists of a player set $N = \{1, 2, \dots, n\}$ and a value function $v : 2^N \rightarrow R$ with $v(\emptyset) = 0$. $\forall S \subseteq N$, $v(S)$ represents the profit obtained by S without the help of others.

A game $\Gamma = (N, v)$ is *monotone* if $v(S') \leq v(S)$ whenever $S' \subseteq S$. A simple game is a monotonic game with $v : 2^N \rightarrow \{0, 1\}$ such that $v(\emptyset) = 0$ and $v(N) = 1$. A coalition $S \subseteq N$ is *winning* if $v(S) = 1$, and *losing* if $v(S) = 0$. A player i is called a *veto* player if he belongs to all winning coalitions. It is easy to see that i is a veto player if and only if $v(N) = 1$ but $v(N \setminus \{i\}) = 0$.

In cooperative games, we focus on how to distribute the total profit $v(N)$ in a fair or stable way. Different requirements on fairness and stability lead to different kinds of distributions, which are generally referred to *solution concepts*. In this paper, we focus on three solutions based on stability: the *core*, the *least-core* and the *nucleolus*.

Given a cooperative game $\Gamma = (N, v)$, we use $x = (x_1, x_2, \dots, x_n)$ to represent the payoff vector while x_i is the payoff for player i . For convenience, let $x(S) \triangleq \sum_{i \in S} x_i$. We say a payoff vector x is an *imputation* of Γ , if $x(N) = v(N)$ and $\forall i \in N$, $x_i \geq v(\{i\})$ (individual rationality). The *core* of Γ is defined as:

$$\mathcal{C}(\Gamma) := \{x \in R^n : x(N) = v(N) \text{ and } x(S) \geq v(S), \forall S \subseteq N\}.$$

A payoff vector in $\mathcal{C}(\Gamma)$ guarantees that any coalition S can not get more profit if it breaks away from the grand coalition. This is called group rationality.

When $\mathcal{C}(\Gamma) = \emptyset$, there is a nature relaxation of the core: the *least-core*. Given $\varepsilon \leq 0$, an imputation x is in the ε -core of Γ , if it satisfies $x(S) \geq v(S) + \varepsilon$ for all $S \subset N$. Let

$$\varepsilon^* := \sup\{\varepsilon | \varepsilon\text{-core of } \Gamma \text{ is nonempty}\}.$$

The ε^* -core is called the least-core of Γ , denoted by $\mathcal{LC}(\Gamma)$, and the value ε^* is called the $\mathcal{LC}(\Gamma)$ value. Obviously, the optimal solution of the following linear program LP_1 is exactly the value and the imputations in $\mathcal{LC}(\Gamma)$:

$$LP_1 : \begin{array}{ll} \max & \varepsilon \\ \text{s.t.} & \begin{cases} x(S) \geq v(S) + \varepsilon, & \forall S \subset N \\ x_i \geq v(\{i\}), & \forall i \in N \\ x(N) = v(N). \end{cases} \end{array}$$

Now we turn to the concept of the *nucleolus*. Given any imputation x , the excess of a coalition S under x is defined as $e(x, S) = x(S) - v(S)$, which can be viewed as the satisfaction degree of the coalition S under the given x . The *excess vector* is the vector $\theta(x) = (e(x, S_1), e(x, S_2), \dots, e(x, S_{2^n-2}))$, where S_1, \dots, S_{2^n-2} is a list of all nontrivial subsets of N that satisfies $e(x, S_1) \leq e(x, S_2) \leq \dots \leq e(x, S_{2^n-2})$. The nucleolus of the game Γ , denoted by $\eta(\Gamma)$, is the imputation x that lexicographically maximizes the excess vector $\theta(x)$.

Kopelowitz [15], as well as Maschler, Peleg and Shapley [16] proposed that $\eta(\Gamma)$ can be computed by recursively solving the following sequential linear programs $SLP(\eta(\Gamma))$ ($k = 1, 2, \dots$):

$$LP_k : \begin{array}{ll} \max & \varepsilon \\ \text{s.t.} & \begin{cases} x(S) = v(S) + \varepsilon_r, & \forall S \in \mathcal{J}_r, \quad r = 0, 1, \dots, k-1 \\ x(S) \geq v(S) + \varepsilon, & \forall S \in 2^N \setminus \bigcup_{r=0}^{k-1} \mathcal{J}_r \\ x_i \geq v(\{i\}), & \forall i \in N \\ x(N) = v(N). \end{cases} \end{array}$$

Initially, we set $\mathcal{J}_0 = \{\emptyset, N\}$ and $\varepsilon_0 = 0$. The number ε_r is the optimal value of the r -th program LP_r , and $\mathcal{J}_r = \{S \subseteq N : x(S) = v(S) + \varepsilon_r, \forall x \in X_r\}$, where $X_r = \{x \in R^n : (x, \varepsilon_r) \text{ is an optimal solution of } LP_r\}$. We call a coalition in \mathcal{J}_r *fixed* since its allocation is fixed to a number. Kopelowitz [15] showed that this procedure converges in at most n steps. Moreover, the nucleolus always exists and it is unique. When $\mathcal{C}(\Gamma) \neq \emptyset$, $\eta(\Gamma) \in \mathcal{C}(\Gamma)$; and $\eta(\Gamma) \in \mathcal{LC}(\Gamma)$, otherwise.

Proposition 1 [10, 18] *A simple game $\Gamma = (N, v)$ has a nonempty core if and only if there exists a veto player. Moreover,*

- (1) $x \in \mathcal{C}(\Gamma)$ if and only if $x_i = 0$ for each $i \in N$ who is not a veto player;
- (2) when $\mathcal{C}(\Gamma) \neq \emptyset$, the nucleolus of Γ is given by $x_i = \frac{1}{k}$ if i is a veto player and $x_i = 0$ otherwise, where k is the number of veto players.

2.2 Threshold Matching Games

We now introduce the definitions of threshold matching games. For more detailed introduction, please refer to [1, 10, 14].

Given a graph $G = (V, E)$, a matching M is a set of edges that no two edges in M have a vertex in common. The size of M is denoted by $|M|$. A matching is maximum if its size is maximum among all matchings in G . When

there is a weight associated with each edge $w : E \rightarrow R^+$, a matching M is called a maximum weight matching if its weight $w(M) = \sum_{e \in M} w(e)$ is maximum among all matchings.

For a weighted graph $G = (V, E; w)$ and a threshold $T \in R^+$, the corresponding *threshold matching game* (TMG) is a cooperative game defined as $\Gamma = (V, w; T)$. We have the player set V and $\forall S \subseteq V$,

$$v(S) \triangleq \begin{cases} 1, & \text{if } w(M) \geq T, \text{ where } M \text{ is the maximum weight matching of } G[S] \\ 0, & \text{otherwise} \end{cases}$$

where $G[S]$ is the induced subgraph by S on G .

Obviously, TMG is a simple game, and a player i is a veto player if and only if the weight of the maximum weight matching in $G[V \setminus \{i\}]$ is less than T . By Proposition 1, when there is a veto player, the core and the nucleolus can be given directly. However, when the core is empty, the least-core and the nucleolus is hard to compute [1].

Proposition 2 [1] *Computing the least-core and nucleolus of TMG is NP-hard if the core of the TMG is empty.*

In the following we restrict ourselves to *threshold cardinality matching game* (TCMG) $\Gamma = (V; T)$ based on unweighted graph $G = (V, E)$. That is, $\forall S \subseteq V$, $v(S) = 1$ if the size of a maximum matching in $G[S]$ is no less than T , and $v(S) = 0$ otherwise.

Throughout the rest of this paper, we use the following notations:

- \mathcal{M} : the set of matchings of G ;
- \mathcal{M}_T : the set of matchings M_T of G whose sizes are exact T , we call $M_T \in \mathcal{M}_T$ a minimal winning coalition;
- \mathcal{M}^* : the set of matchings M^* of G with maximum size;
- v^* : the size of the maximum matching of G ;
- n : $n = |V|$ is the number of players.

Let $G' = (V', E')$ be a subgraph of G . We usually use $i \in G'$ instead of $i \in V'$, and $G' \setminus \{j\}$ instead of $G'[V' \setminus \{j\}]$. For any imputation x , we define $x(G') \triangleq \sum_{i \in V'} x_i$ and $x(G'_{-j}) \triangleq x(G' \setminus \{j\}) = \sum_{i \in V', i \neq j} x_i$. Specially, $\forall e = (i, j) \in E$, we let $x(e) = x(\{i, j\}) = x_i + x_j$; and $\forall M \in \mathcal{M}$, we let $x(M) = \sum_{e \in M} x(e)$.

Before entering into the details, we begin with Gallai-Edmonds Decomposition of a graph, which plays an important role in the nucleolus characterization.

2.3 Gallai-Edmonds Decomposition

Let $G = (V, E)$ be a graph. A matching of G is called a *perfect matching* if it covers all vertices of G , and a *nearly perfect matching* if it covers all vertices except one. G is called *factor-critical* if removing any vertex of G , the rest graph has a perfect matching.

Given $A \subseteq V$, let $G \setminus A$ denote the subgraph induced by vertices $V \setminus A$. $G \setminus A$ is composed of one or several maximal connected components (hereinafter

referred to as components). A component of $G \setminus A$ is called *even* (*odd*) if it contains even (odd) number of vertices. Denote by $\mathcal{B} = \mathcal{B}(A)$ and $\mathcal{D} = \mathcal{D}(A)$ the set of even components and odd components in $G \setminus A$, respectively. We use $V(\mathcal{B})$ ($V(\mathcal{D})$) to represent all vertices in even (odd) components. A set $A \subseteq V$ is called a *Tutte set* if each maximum matching M^* of G can be decomposed as $M^* = M_{\mathcal{B}} \cup M_{A,\mathcal{D}} \cup M_{\mathcal{D}}$, where $M_{\mathcal{B}}$ induces a perfect matching in any even component $B \in \mathcal{B}$, $M_{\mathcal{D}}$ induces a nearly perfect matching in any odd component $D \in \mathcal{D}$, and $M_{A,\mathcal{D}}$ is a matching which matches every vertex in A to some vertex in an odd component in \mathcal{D} . Thus, if A is a Tutte set, the size v^* of a maximum matching in G satisfies

$$v^* = \sum_{B \in \mathcal{B}} \frac{|B|}{2} + |A| + \sum_{D \in \mathcal{D}} \frac{(|D| - 1)}{2}.$$

Lemma 1 (Gallai-Edmonds Decomposition) [14,19,23] *Given $G = (V, E)$, one can construct a Tutte set $A \subseteq V$ in polynomial time such that*

1. *all odd components $D \in \mathcal{D}$ are factor-critical;*
2. *$\forall D \in \mathcal{D}$ there is a maximum matching M^* of G which does not completely cover D (we say M^* leaves D uncovered).*

In the following we assume that $A \subseteq V$ is a fixed Tutte set satisfying the condition 1 and 2 in Lemma 1. Thus, for such a Tutte set A , we have the following facts:

- 1) For any $D \in \mathcal{D}$ and any maximum matching M^* , M^* matches at most one vertex in A to D , hence, $|A| \leq |\mathcal{D}|$. And when $\mathcal{D} \neq \emptyset$, $|A| < |\mathcal{D}|$.
- 2) For any Tutte set A satisfying conditions in Lemma 1, the size of $A \cup V(\mathcal{B})$ is fixed which does not change with the different choice of A .

3 Least-core of TCMG

In this section we firstly show that the least-core problem for TCMG is much easier than that for TMG by providing a polynomial time algorithm. Besides, we introduce a non-cooperative two-player zero-sum game, called matching intercept game. There is a close relationship between the mixed Nash equilibrium of this game and the least-core of TCMG.

Throughout this section, let $\Gamma = (V; T)$ be the TCMG defined on an unweighted graph $G = (V, E)$ with threshold $T : 1 \leq T \leq v^*$. Since both testing the core nonemptiness and finding a core member can be done efficiently, we focus on the case where $\mathcal{C}(\Gamma) = \emptyset$, i.e., there is no veto players in Γ .

3.1 Least-core

For a TCMG $\Gamma = (V; T)$, denote $\mathcal{E}(\Gamma) = \{\{i\} : i \in V\} \cup \{\{M_T : M_T \in \mathcal{M}_T\}\} \cup \{V\}$. We call a coalition $S \in \mathcal{E}(\Gamma)$ *essential coalition*. We can show that

for Γ , the least-core and nucleolus can be determined completely by the essential coalitions.

Suppose $S \subseteq V$ is a coalition of Γ . If S wins, then S contains a minimal winning coalition $M_T \in \mathcal{M}_T$, and

$$x(S) - v(S) = x(M_T) - 1 + \sum_{i \in S \setminus V(M_T)} x_i \geq x(M_T) - 1.$$

Since $x_i \geq 0$ for all $i \in V$, S cannot be fixed before M_T or any $i \in S \setminus V(M_T)$. After M_T and all $i \in S \setminus V(M_T)$ are fixed, S is also fixed, i.e. coalitions like S are redundant; If S loses, we can decompose S in the following way:

$$x(S) - v(S) = \sum_{i \in V(S)} x_i \geq x_i, i \in S.$$

We can conclude that S cannot be fixed before any $i \in V(S)$ through the same analysis. When all $i \in V(S)$ are fixed, S is fixed, i.e. S is also redundant in this case.

From above analysis, we can conclude that the least-core $\mathcal{LC}(\Gamma)$ of TCMG can be characterized as the optimal solution of the following linear program LP_1^T :

$$LP_1^T : \begin{array}{ll} \max & \varepsilon \\ \text{s.t.} & \begin{cases} x(M_T) \geq 1 + \varepsilon, & \forall M_T \in \mathcal{M}_T \\ x(V) = 1 \\ x_i \geq 0, & i = 1, 2, \dots, n. \end{cases} \end{array}$$

We can obtain the same result in terms of the nucleolus, i.e., we have the following proposition.

Proposition 3 *Dropping the constraints associated with all $S \notin \mathcal{E}(\Gamma)$ will not change the result of LP_1 and $SLP(\eta(\Gamma))$.*

Therefore, when the threshold T is a fixed number independent of the input size, the size of the linear programs in LP_1 and $SLP(\eta(\Gamma))$ are all polynomial. It follows that the least-core and the nucleolus can be computed efficiently. However, when the threshold T is relevant to the input size, the difficulty on computing is that each linear program in LP_1 and $SLP(\eta(\Gamma))$ remains exponential size and we lack the understanding of the composition of \mathcal{J}_r . In the following sections, we assume the threshold T is relevant to the input size.

We firstly show that least-core can be solved efficiently by ellipsoid method with a polynomial time separation oracle.

Theorem 1 *Let $\Gamma = (V; T)$ be a TCMG with empty core. Then the problems of computing the $\mathcal{LC}(\Gamma)$ value, finding a $\mathcal{LC}(\Gamma)$ member and checking if an imputation is in $\mathcal{LC}(\Gamma)$ are all polynomial time solvable.*

Proof. A polynomial time separation oracle for LP_1^T is as follows. Let (x, ε) be a candidate solution for LP_1^T . Setting edge cost $\tilde{c}(e) = x_i + x_j$ ($\forall e = (i, j) \in E$) on

the edge set of graph $G = (V, E)$. Then we compute the minimum cost matching M of size T . If $x(M) \geq 1 + \varepsilon$, then (x, ε) is a feasible solution; otherwise, the inequality $x(M) \geq 1 + \varepsilon$ is a violated constraint. Here, computing the minimum cost matching of fixed size can be done in polynomial time. \square

The above theorem shows that finding an imputation in the least-core of TCMG is not as hard as that of the general case TMG.

In the following, we further provide a characterization of the least core of TCMG under some conditions. Denote $\mathcal{M}_T = \{M_1, M_2, \dots, M_m\}$ be the set of all matchings whose sizes are exact T , and let a_1, a_2, \dots, a_m be the indicator vectors of the matchings in \mathcal{M}_T . Consider the dual program of LP_1^T :

$$DLP_1^T : \begin{array}{ll} \min & \delta - 1 \\ \text{s.t.} & \begin{cases} \sum_{M_j: i \in M_j} y_j \leq \delta, & i = 1, 2, \dots, n \\ \sum_{j=1}^m y_j = 1 \\ y_j \geq 0, & j = 1, 2, \dots, m. \end{cases} \end{array}$$

Followed from the duality theorem, DLP_1^T has the same optimal value as LP_1^T . Hence, we have the following result which is quite useful in the algorithm design for the nucleolus in next sections.

Theorem 2 *Let $\Gamma = (V; T)$ be a TCMG with empty core. If $(\frac{2T}{n}, \dots, \frac{2T}{n})_n$ is a convex combination of a_1, a_2, \dots, a_m , then*

1. *the value of $\mathcal{LC}(\Gamma)$ is $\varepsilon = \frac{2T}{n} - 1$;*
2. *$(\frac{1}{n}, \dots, \frac{1}{n})_n \in \mathcal{LC}(\Gamma)$;*
3. *if there exists a convex combination and the coefficient corresponding to a_i is strictly greater than zero, we have that the i -th constraint belongs to \mathcal{J}_1 in $SLP(\eta(\Gamma))$.*

Proof. The condition $(\frac{2T}{n}, \dots, \frac{2T}{n})_n$ is the convex combination of a_j ($j = 1, 2, \dots, m$) is equivalent to the fact that there is a feasible solution to DLP_1^T with the objective function value being $\frac{2T}{n} - 1$. On the other hand, it is easy to check $(x, \varepsilon) = (\frac{1}{n}, \dots, \frac{1}{n}, \frac{2T}{n} - 1)$ is feasible to LP_1^T . Followed from duality theorem of LP, $(x, \varepsilon) = (\frac{1}{n}, \dots, \frac{1}{n}, \frac{2T}{n} - 1)$ is an optimal solution of LP_1^T , yielding that $(\frac{1}{n}, \dots, \frac{1}{n})_n \in \mathcal{LC}(\Gamma)$. By Complementary Slackness Theorem, we know if $a_i > 0$, the i -th constraint is tight in each optimal solution. Thus, it belongs to \mathcal{J}_1 in $SLP(\eta(\Gamma))$. \square

3.2 Matching Intercept Games

Now we define a non-cooperative zero-sum game on graph $G = (V, E)$ with two players, the *interceptor* and the *matcher*. The pure strategy set of the interceptor is V and the pure strategy set of the matcher is \mathcal{M}_T . If the vertex chosen by the interceptor intersects with the matching chosen by the matcher, then the

interceptor wins and gets payoff 1; otherwise, he loses and gets payoff 0. We call this non-cooperative game *matching intercept game*(MIG).

For the notion of *mixed Nash Equilibrium*, players select strategies at random and act to maximum their own expected profit. In MIG, let $p = (p_1, p_2, \dots, p_n)$ be the interceptor's probability distribution over his pure strategies, then based on Maxmin Theorem, the optimal solution of the following linear program gives the mixed Nash Equilibrium of MIG:

$$\begin{aligned} \widetilde{LP}_1^T : \quad & \max \quad \alpha \\ & \text{s.t.} \quad \begin{cases} p(M_T) \geq \alpha, & \forall M_T \in \mathcal{M}_T \\ p(V) = 1 \\ p_i \geq 0, & i = 1, 2, \dots, n. \end{cases} \end{aligned}$$

It is obvious that linear program \widetilde{LP}_1^T is equivalent to linear program LP_1^T . Thus, for TCMG and MIG defined on the same graph G , the least-core of TCMG is the same as the mixed Nash Equilibrium of MIG.

In the following sections, we focus on the following three typical cases of TCMG to give a clearer characterization on the nucleolus:

- Edge Coalitional Games (ECG): $T = 1$ (section 4);
- TCMG on graphs with a perfect matching (section 5);
- TCMG on bipartite graphs (section 6).

We aim to show that $\eta(\Gamma)$ is completely determined by edge coalitions (Coalition which contains exact two vertices and these two vertices form an edge in G is called edge coalition.), single player coalitions (which contains only one player) as well as grand coalition rather than essential coalitions.

4 Edge Coalitional Games (T=1)

In this section, we consider the edge coalitional game (ECG) $\Gamma^1 = (V; 1)$ defined on an unweighted graph $G = (V, E)$, i.e., the TCMG with threshold $T = 1$. If there is a 0-degree vertex in G , then it has no contribution to any coalitions, i.e., its allocation must be 0. In the following, we assume that there is no 0-degree vertices in G . It is easy to see that $\mathcal{C}(\Gamma^1) \neq \emptyset$ if and only if there exists a vertex $i \in V$ such that there is no edges in $G \setminus \{i\}$, that is, the graph G is a star-like graph.

When $\mathcal{C}(\Gamma^1) = \emptyset$, the linear program for $\mathcal{LC}(\Gamma^1)$ is as follows:

$$LP_1^1 : \quad \begin{aligned} & \max \quad \varepsilon \\ & \text{s.t.} \quad \begin{cases} x_i + x_j \geq 1 + \varepsilon, & \forall e = (i, j) \in E \\ x(V) = 1 \\ x_i \geq 0, & i = 1, 2, \dots, n. \end{cases} \end{aligned}$$

According to Gallai-Edmonds Decomposition, every graph can be decomposed into A , \mathcal{B} , \mathcal{D} . Let \mathcal{D}_0 be the set of singletons in \mathcal{D} (\mathcal{D}_0 may be empty). Since \mathcal{D}_0 is the set of singletons, we ambiguously use \mathcal{D}_0 instead of $V(\mathcal{D}_0)$. Let

G_0 be a bipartite graph with vertex set $A \cup \mathcal{D}_0$ and edge set consisting of edges with two endpoints in A and \mathcal{D}_0 separately. Find a maximum matching M_0 in G_0 . Denote the matched vertices in A and \mathcal{D}_0 by A_1 and \mathcal{D}_{01} with respect to M_0 . Let $A_2 = A \setminus A_1$ and $\mathcal{D}_{02} = \mathcal{D} \setminus \mathcal{D}_{01}$.

We firstly consider the simple case that $\mathcal{D}_{02} = \emptyset$, that is, all singletons in \mathcal{D} can be matched to A . By making use of Theorem 2, we show that in this case, the least-core value and an imputation in the least-core can be obtained directly. Then we generalize the result into the case $\mathcal{D}_{02} \neq \emptyset$.

Theorem 3 *Given an ECG $\Gamma^1 = (V; 1)$, if $\mathcal{D}_{02} = \emptyset$, then the value of $\mathcal{LC}(\Gamma^1)$ is $\varepsilon = \frac{2}{n} - 1$ and $(\frac{1}{n}, \dots, \frac{1}{n})_n \in \mathcal{LC}(\Gamma)$.*

Proof. By Theorem 2, it is enough to show that $(\frac{2}{n}, \dots, \frac{2}{n})$ is the convex combination of the coefficients of $x_i + x_j \geq 1 + \varepsilon, e = (i, j) \in E$.

Since $\mathcal{D}_{02} = \emptyset$, let M be a maximum matching in G such that M matches all \mathcal{D}_0 into A . Then $\mathcal{D}_0 \subseteq \mathcal{D}_1 \triangleq \{D \in \mathcal{D} : D \text{ is covered by } M\}$ and $\mathcal{D}_2 \triangleq \mathcal{D} \setminus \mathcal{D}_1$ is the set of uncovered factor-critical graphs. So for any $D_i \in \mathcal{D}_2$, we have $n_i = |D_i| \geq 3$. Let \tilde{M} is the induced matching from M in $V(\mathcal{B}) \cup A \cup V(\mathcal{D}_1)$. Then \tilde{M} is a perfect matching in $G[V(\mathcal{B}) \cup A \cup V(\mathcal{D}_1)]$. For any edge $e \in \tilde{M}$, we set $\frac{2}{n}$ to the element in the convex combination corresponding to this constraint. There are in total $\frac{|V(\mathcal{B}) \cup A \cup V(\mathcal{D}_1)|}{2}$ constraints, and each vertex in $V(\mathcal{B}) \cup A \cup V(\mathcal{D}_1)$ appears exactly once in these constraints. For any $D_i \in \mathcal{D}_2$, since it is factor-critical, then for every vertex $k \in V(D_i)$, there exists a perfect matching M_k in $G[V(D_i) \setminus \{k\}]$. For any edge $e \in M_k$, we set $\frac{2}{(n_i-1)n}$ to the element in the convex combination corresponding to this constraint. When we traverse all vertices $k \in V(D_i)$, there are in total $\frac{n_i(n_i-1)}{2}$ constraints, and for each vertex k , it appears in exactly $n_i - 1$ constraints. Set the elements in the convex combination corresponding to all other constraints to be 0. It is easy to check the convex combination of these constraints is $(\frac{2}{n}, \dots, \frac{2}{n})$. This finishes our proof. \square

If $\mathcal{D}_{02} \neq \emptyset$, we cannot find such a convex combination since there is no edges in \mathcal{D}_{02} . But if we delete \mathcal{D}_{02} from G , we can find a convex combination in $G' = G[V \setminus \mathcal{D}_{02}]$ by using the similar argument in the proof of Theorem 3. Denote Γ' to be the corresponding ECG defined on G' . Then the value of $\mathcal{LC}(\Gamma')$ is $\frac{2}{n'} - 1$ where $n' = n - |\mathcal{D}_{02}|$. We can know $\frac{2}{n'} - 1$ is an upper bound of the value of $\mathcal{LC}(\Gamma^1)$. If we can find a feasible solution (\tilde{x}, ε) of the original game with $\varepsilon = \frac{2}{n'} - 1$, then the value of $\mathcal{LC}(\Gamma^1)$ is also $\frac{2}{n'} - 1$. Consider the following imputation:

$$\tilde{x}_i = \begin{cases} \frac{1}{n'}, & i \in V(\mathcal{B}) \\ \frac{2}{n'}, & i \in V(A) \text{ and } \mathcal{D}_{02} \rightarrow i \\ \frac{1}{n'}, & i \in V(A) \text{ and } \mathcal{D}_{02} \nrightarrow i \\ 0, & i \in V(\mathcal{D}_{01}) \text{ and } \mathcal{D}_{02} \rightarrow i \\ \frac{1}{n'}, & i \in V(\mathcal{D}_{01}) \text{ and } \mathcal{D}_{02} \nrightarrow i \\ 0, & i \in V(\mathcal{D}_{02}). \end{cases}$$

Here, $\mathcal{D}_{02} \rightarrow i$ represents i is reachable from \mathcal{D}_{02} by M_0 -alternating path in G_0 ; $\mathcal{D}_{02} \nrightarrow i$ represents i is unreachable from \mathcal{D}_{02} by M_0 -alternating path in G_0 .

We can easily check that the imputation \tilde{x} with $\varepsilon = \frac{2}{n'} - 1$ is feasible. Otherwise, if there is an edge $e \in E$ such that $\tilde{x}(e) < \frac{2}{n'}$, e must be between a vertex $i \in \mathcal{D}_{01}$ (and $\mathcal{D}_{02} \rightarrow i$) and a vertex in A_2 or between a vertex in \mathcal{D}_{02} and a vertex in A_2 . But there exists a M_0 -augmenting path in G_0 in these two cases, contradicting to M is a maximum matching in G_0 . Hence, the value of $\mathcal{LC}(\Gamma^1)$ is $\frac{2}{n'} - 1$.

We then focus on the computing of nucleolus. Since we have seen $\varepsilon_1 = \frac{2}{n'} - 1$, we can prove LP_k^1 in $SLP(\eta(\Gamma^1))$ can be rewritten as:

$$LP_k^1 : \begin{array}{ll} \max & \varepsilon \\ \text{s.t.} & \begin{cases} x(e) = \frac{2}{n'} - \varepsilon_1 + \varepsilon_r, & e \in E_r, r = 1, \dots, k-1 \\ x_i = -\varepsilon_1 + \varepsilon_r, & i \in V_r, r = 1, \dots, k-1 \\ x(e) \geq \frac{2}{n'} - \varepsilon_1 + \varepsilon, & e \in E \setminus \bigcup_{r=1}^{k-1} E_r \\ x_i \geq -\varepsilon_1 + \varepsilon, & i \in V \setminus \bigcup_{r=1}^{k-1} V_r \\ x(V) = 1, x_i \geq 0, & i \in V. \end{cases} \end{array}$$

Initially set $E_0 = V_0 = \emptyset$ and $\varepsilon_0 = 0$. The number ε_r is the optimal value of the r -th program LP_r^1 , and $E_r = \{e \in E : x(e) = 1 + \varepsilon_r, \forall x \in X_r\}$, $V_r = \{i \in N : x_i = 1 - \frac{2}{n'} + \varepsilon_r, \forall x \in X_r\}$, where $X_r = \{x \in R^n : (x, \varepsilon_r) \text{ is an optimal solution of } LP_r^1\}$.

In the next sections, we will show that for any threshold, LP_k^T have the same appearance as LP_k^1 under some restrictions.

5 TCMG on Graph with Perfect Matching

Now we consider the general case $\Gamma^T = (V; T)$ with arbitrary threshold $1 \leq T \leq v^*$. We denote the corresponding sequential linear programming as LP_k^T .

In the following theorem, we firstly show that for the graphs with a perfect matching, the least-core of Γ^T is independent of T . Then we use this characterization to prove that the nucleolus of Γ^T can be obtained in polynomial time and $\eta(\Gamma^T)$ is also independent of T , i.e., LP_k^T can be rewritten as:

$$LP_k^T : \begin{array}{ll} \max & \varepsilon^T \\ \text{s.t.} & \begin{cases} x(e) = \frac{2}{n} - \varepsilon_1^T + \varepsilon_r^T, & e \in E_r, r = 1, \dots, k-1 \\ x_i = -\varepsilon_1^T + \varepsilon_r^T, & i \in V_r, r = 1, \dots, k-1 \\ x(e) \geq \frac{2}{n} - \varepsilon_1^T + \varepsilon^T, & e \in E \setminus \bigcup_{r=1}^{k-1} E_r \\ x_i \geq -\varepsilon_1^T + \varepsilon^T, & i \in V \setminus \bigcup_{r=1}^{k-1} V_r \\ x(V) = 1, x_i \geq 0, & i \in V. \end{cases} \end{array} \quad (1)$$

Initially, we set $E_0 = V_0 = \emptyset$ and $\varepsilon_0^T = 0$. The number ε_r^T is the optimal value of the r -th program LP_r^T , and $E_r = \{e \in E : x(e) = 1 + \varepsilon_r^T, \forall x \in X_r\}$, $V_r = \{i \in N : x_i = 1 - \frac{2}{n} + \varepsilon_r^T, \forall x \in X_r\}$, where $X_r = \{x \in R^n : (x, \varepsilon_r^T) \text{ is an optimal solution of } LP_r^T\}$.

Theorem 4 Suppose $G = (V, E)$ is a simple graph which has a perfect matching and $\Gamma^T = (V; T)$ is a TCMG defined on G . Let $\Gamma^1 = (V; 1)$ be the corresponding ECG defined also on G . Then

1. the value of $\mathcal{LC}(\Gamma^T)$ is $\varepsilon_1^T = \frac{2T}{n} - 1$;
2. $\mathcal{LC}(\Gamma^T) = \mathcal{LC}(\Gamma^1)$;
3. $\eta(\Gamma^T) = \eta(\Gamma^1)$.

Proof.

1. By Theorem 2, it is enough to show that $(\frac{2T}{n}, \dots, \frac{2T}{n})$ is the convex combination of the coefficients of $x(M_T) \geq 1 + \varepsilon, M_T \in \mathcal{M}_T$. Since G has a perfect matching M^* , then $v^* = \frac{n}{2}$. Without loss of generality, we reset the labels of players as $(1, 2), \dots, (n-1, n)$, here (i, j) means it is a matching edge in M^* . Then there are constraints like the following in LP_1^T

$$\begin{aligned} (x_1 + x_2) + \dots + (x_{2T-1} + x_{2T}) &\geq 1 + \varepsilon^T \\ (x_3 + x_4) + \dots + (x_{2T+1} + x_{2T+2}) &\geq 1 + \varepsilon^T \\ \dots & \\ (x_{n-1} + x_n) + \dots + (x_{2T-3} + x_{2T-2}) &\geq 1 + \varepsilon^T. \end{aligned} \tag{2}$$

We put all the constraints above into the combination with an element $\frac{2}{n}$. There are in total $\frac{n}{2}$ constraints, and each vertex in V appears exactly T times in these constraints. It is easy to check this convex combination of these constraints is $(\frac{2T}{n}, \dots, \frac{2T}{n})$.

2. Suppose $x = (x_1, \dots, x_n)$ with $\varepsilon^T = \frac{2T}{n} - 1$ is an optimal solution in LP_1^T (the first linear program in $SLP(\eta(\Gamma^T))$), $e = (i, j)$ is a maximum matching edge in M^* . Since constraints in (2) are fixed, then

$$x_i + x_j = \frac{2}{n}, \quad \forall e = (i, j) \in M^*. \tag{3}$$

If $e = (i, j)$ is not an edge of M^* , e must be in a matching M' with size $v^* - 1$ and all the other $v^* - 2$ edges except e are belonging to M^* , i.e., those edges are fixed to $\frac{2}{n}$. Since for all $M_T \in \mathcal{M}_T$, $x(M_T) \geq 1 + \varepsilon^T = \frac{2T}{n}$, we have

$$x_i + x_j \geq \frac{2}{n}, \quad \forall e = (i, j) \in E \setminus M^*. \tag{4}$$

From (3) and (4), we can see that $(x, \frac{2}{n} - 1)$ is also an optimal solution of LP_1^1 .

On the other hand, let $x' = (x'_1, \dots, x'_n)$ with $\varepsilon_1^1 = \frac{2}{n} - 1$ be an optimal solution in LP_1^1 . We can quickly check x' with $\varepsilon_T = \frac{2T}{n} - 1$ is an optimal solution in LP_1^T .

Therefore, $\mathcal{LC}(\Gamma^T) = \mathcal{LC}(\Gamma)$.

3. Suppose the set of fixed constraints in LP_1^T is $\mathcal{M}'_T \subseteq \mathcal{M}_T$ (and the set of fixed coalitions is also \mathcal{M}'_T) and E_1 is the fixed constraints of LP_1^1 . Firstly, we prove that $E_T = E_1$, $E_T = \{e | e \in M_T \in \mathcal{M}'_T\}$.

Since the system of linear equations $x(M_T) = \frac{2T}{n}$, $M_T \in \mathcal{M}'_T$ is equivalent to the system of linear equations $x(e) = \frac{2}{n}$, $e \in E_T$. Otherwise, if there exists some $e \in M_T \in \mathcal{M}'_T$ with $x(e) > \frac{2}{n}$, $x(M_T)$ cannot equal to $\frac{2T}{n}$, due to $x(e) \geq \frac{2}{n}$, $\forall e \in E$. Because $\mathcal{LC}(\Gamma^T) = \mathcal{LC}(\Gamma)$, $E_{\mathcal{T}} = E_1$.

Then we simplify the sequence of linear programs.

Case 1: Consider the winning constraints like

$$x(S) = \sum_{e \in E'} x(e) + \sum_{e \in E''} x(e) + \sum_{i \in V'} x_i \geq 1 + \varepsilon^T.$$

Here, $E' \subseteq E$, $E' \cap E_1 = \emptyset$, $E'' \subseteq E_1$ and $V' \subseteq V$. The size of the maximum matching in $S = V(E') \cup V(E'') \cup V'$ is not less than T and suppose $|E'| \geq 2$. It will be fixed after any

$$x(e) + \sum_{e \in S' \subseteq E_1, |S'|=T-1} x(e) \geq 1 + \varepsilon^T, \quad e \in E' \quad (5)$$

and

$$x_i + \sum_{e \in S'' \subseteq E_1, |S''|=T} x(e) \geq 1 + \varepsilon^T, \quad i \in V', \quad (6)$$

since the excess of the coalition S is not less than any coalition $\{e\} \cup S'$ or $\{i\} \cup S''$. Moreover, it will be fixed automatically after all constraints like (5) and (6) get fixed.

Due to result 1 and 2 above, we can rewrite (5) and (6) as

$$x(e) \geq \frac{2}{n} - \varepsilon_1^T + \varepsilon^T, \quad e \in E \setminus E_1, \quad (7)$$

$$x_i \geq -\varepsilon_1^T + \varepsilon^T, \quad i \in V. \quad (8)$$

Case 2: Consider the losing constraints like

$$x(S) = \sum_{e \notin E_1} x(e) + \sum_{e \in E_1} x(e) + \sum x_i \geq \varepsilon^T.$$

The maximum matching in S is less than T and it will be fixed after (7) and (8), since the excess of the coalition S is not less than any subset in S . Moreover, it will be fixed automatically after all constraints like (7) and (8) get fixed.

Hence, the sequential linear programs LP_k^T of $\eta(\Gamma^T)$ can be rewritten as linear program (1). It is obvious that the optimal solutions of LP_k^T and LP_k^1 are the same except $\varepsilon_k^T = \varepsilon_k^1 + \frac{2(T-1)}{n}$ before ε_k gets positive, i.e., $\eta(\Gamma^T) = \eta(\Gamma)$.

This finishes our proof. \square

6 TCMG on Bipartite Graphs

For bipartite graphs, we can obtain the similar result as Theorem 4. Let $G = (L, R; E)$ be a bipartite graph with vertex set $L \cup R$ and edge set E . Find a

maximum matching M^* in G . Denote the matched vertices in L and R as L_1 and R_1 with respect to M^* respectively. Let $L_2 = L \setminus L_1$ and $R_2 = R \setminus R_1$.

If both L_2 and R_2 are empty, it is reduced to the situation in section 5. So we assume at least one of L_2 and R_2 is not empty.

If we delete L_2 and R_2 from G , we can find the least-core value and an imputation in least-core by Theorem 4. Denote Γ' to be the corresponding TCMG defined on G' where G' is the induced subgraph by $(L \cup R) \setminus (L_2 \cup R_2)$ in G . Then the value of $\mathcal{LC}(\Gamma')$ is $\frac{2T}{n'} - 1$ where $n' = n - |L_2| - |R_2|$. It is obvious that $\frac{2T}{n'} - 1$ is an upper bound of the value of $\mathcal{LC}(\Gamma^T)$. We then show that this is actually the value of the least-core in the bipartite graphs.

Theorem 5 *Suppose $G = (L, R; E)$ is a bipartite graph and $\Gamma^T = (V; T)$ is a TCMG defined on G . Let $\Gamma^1 = (V; 1)$ be the corresponding ECG defined also on G . Then*

1. *the value of $\mathcal{LC}(\Gamma^T)$ is $\varepsilon_1^T = \frac{2T}{n'} - 1$;*
2. *$\mathcal{LC}(\Gamma^T) = \mathcal{LC}(\Gamma^1)$;*
3. *$\eta(\Gamma^T) = \eta(\Gamma^1)$.*

Here $n' = n - |L_2| - |R_2|$.

Proof. We only prove the first result. The second and the third ones are the same as Theorem 4.

If we can find a feasible solution (\tilde{x}, ε) of the original game with $\varepsilon = \frac{2T}{n'} - 1$, then the value of $\mathcal{LC}(\Gamma^1)$ is also $\frac{2T}{n'} - 1$. Here, we use $L_2 \rightarrow i, i \in L_1 \cup R_1$ to represent that i is reachable from L_2 by M^* -alternating path in G . We denote these vertices which are in L_1 by L_{11} and the vertices which are in R_1 by R_{11} ; Similarly, $R_2 \rightarrow i, i \in L_1 \cup R_1$ represents i is reachable from R_2 by M^* -alternating path in G , and we denote these vertices which are in L_1 by L_{12} and the vertices which are in R_1 by R_{12} . Denote $L_{13} = L_1 \setminus (L_{11} \cup L_{12})$, $R_{13} = R_1 \setminus (R_{11} \cup R_{12})$. Note that $L_{11} \cap L_{12} = \emptyset$ and $R_{11} \cap R_{12} = \emptyset$. Otherwise, without loss of generality, we assume $k \in L_{11} \cap L_{12}$. Then, there exists two M^* -alternating paths in G , $P_1 : l_2 \rightarrow k, l_2 \in L_2$ and $P_2 : r_2 \rightarrow k, r_2 \in R_2$. Find k' to be the first intersection point of P_1 and P_2 . Here first intersection point means there is no other intersection points locating in $l_2 \xrightarrow{P_1} k'$ or $r_2 \xrightarrow{P_2} k'$. We know that one of the two edges incidence of k' in P_1 and P_2 is matched edge and the other is unmatched. Then we find a M^* -augmenting path in G : $l_2 \xrightarrow{P_1} k' \xrightarrow{P_2} r_2$, contradicting to M^* is a maximum matching. Hence such k does not exist.

Consider the following imputation:

$$\tilde{x}_i = \begin{cases} 0, & i \in L_2 \cup R_2 \\ \frac{2}{n'}, & i \in R_{11} \cup L_{12} \\ 0, & i \in L_{11} \cup R_{12} \\ \frac{1}{n'}, & i \in L_{13} \cup R_{13} \end{cases}$$

Therefore our imputation \tilde{x} is well defined. Then we can easily check that the imputation \tilde{x} with $\varepsilon = \frac{2T}{n'} - 1$ is feasible like section 4. In fact, we can find

the union of L_{11} and R_{12} (or L_{12} and R_{11}) is a Tutte set in Gallai-Edmonds Decomposition. \square

7 Conclusion

In this paper, we first design a polynomial time algorithm to compute the least-core for threshold cardinality matching games. Based on a general characterization of the least-core for TCMG, we show that computing the nucleolus can be done efficiently for TCMGs defined on graphs with a perfect matching and bipartite graphs.

We conjecture that the ideas behind these results can be generalized to compute the nucleolus of TCMGs defined on general graphs. Another interesting direction is to understand how far can these techniques be extended to the computation of the least-core and the nucleolus of other threshold versions of cooperative games [1].

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